Experimental realization of an approximate transpose operation for qutrit systems using structural physical approximation

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An approximate transpose has been realized for qubit states. In this work, we report experimental implementation of an approximate transpose for three-level systems (qutrit) using the method known as the structural physical approximation. The operation is implemented for the qutrit state encoded in the single photons polarization and path, using measurement and preparation of quantum states.

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In quantum information theory, transpose has a great importance since it can be used for detecting useful entanglement. According to the Peres-Horodecki criteria, separable states give only positive eigenvalues after the partial transposition [1, 2]. However, since the transpose is not a completely positive operation [2], transpose is not a physical operation. This means that the transpose cannot be directly implemented as a legitimate quantum operation in a laboratory for experimental entanglement detection.

In 2002, the structural physical approximation (SPA), a systematic approximation to positive maps, has been proposed for the direct experimental entanglement detection [3]. In this scheme, one admixes a white noise to the original non-physical operations in order to make the approximated operation become completely positive. Since the white noise is isotropic, it does not ruin the properties of the initial operations. The initial proposal of the SPA assumed collective measurements for the spectrum estimation, so quantum memory and coherent quantum operations over multiple identically prepared quantum systems are required. Recently, it has been conjectured that the SPAs to positive maps are entanglement breaking [4], meaning that for those maps that fulfill the conjecture, their SPAs can be realized with the measurement and preparation of quantum states [5]. Hence, the realization of the SPA operation becomes much simpler as it can be implemented as quantum operations over individual quantum systems.

In experimental sides, the SPA of the transpose (SPA-T) operation for photonic qubit systems has been experimentally realized recently using linear optical devices [6]. In Ref. [6], the optimally approximate transpose operation for two-level systems is implemented by the measurement and preparation of the quantum states. Furthermore, the SPA of the partial transpose (SPA-PT) operation for two qubit systems has also been experimentally demonstrated [7]. Since the approximate partial transpose can be decomposed into a convex combination of local operations of individual sub-systems such as the SPA of the transpose and the SPA of the inversion [8], SPA-PT can be realized between two distant parties by local operations and classical channels.

Until now, the experimental implementation of the SPA operation has been considered in only the lowest dimension, i.e., qubit [6]. Since the Peres-Horodecki criteria can be applied to any higher dimensions beyond two qubit systems, SPA-PT can be used for detecting entanglement in high-dimensional systems, such as, $2 \otimes 3$, $2 \otimes 4$, $3 \otimes 3$, etc. Hence, it is required that one can implement SPA operations for qutrit systems in order to realize SPA-PT for higher dimensional states.

In this work, we report experimental realization of the approximate transpose operation for a three-dimensional quantum system (qutrit) using the SPA based on the measurement and preparation of quantum states. The experimental demonstration of the approximate transpose operation is performed in photonic systems, i.e., using single-photon’s polarization and path with linear optical devices. In this SPA scheme for the transpose, the density matrices of quantum states represented in the computational bases $|0\rangle$, $|1\rangle$, and $|2\rangle$ are approximately transposed. The results show that the performance of the realized SPA transpose for qutrit systems is very close to the ideal SPA operation.

Let us begin by briefly introducing the theoretical background of the SPA based on the measurement and preparation of quantum states [4]. The key idea of the SPA is that one admixes a white noise to an original operation $\Lambda$ in order the approximated operation $\tilde{\Lambda}$ to be positive and completely positive, so $\tilde{\Lambda}$ can be implemented in a laboratory. $\tilde{\Lambda}$ can be represented by

$$
\tilde{\Lambda} = (1 - p) \Lambda + pD,
$$

(1)
where $0 \leq p \leq 1$ and the contraction map $D[\rho] = I_d/d$ ($I_d$, $d$ dimensional identity matrix) transforms any quantum states into the maximally mixed state (white noise) and $d$ is the dimension of the quantum states. In addition, the critical $p$ is given by $p = d/(d+1)$ so that the fidelity between $T$ and $\widetilde{T}$ for a pure input state $|\psi\rangle$ is maximal in the approximation, i.e. $F = \text{tr} \left[ T |\psi\rangle \langle \psi| \widetilde{T} |\psi\rangle \langle \psi| \right] = 2/(d+1)$. For the transpose $T$ in 3-dimensional systems (qutrit), the approximated transpose operation $\widetilde{T}$ is given by [4]

\[
\widetilde{T} = \frac{1}{4} T + \frac{3}{4} D.
\]

The SPA to the transpose for the initial qutrit states $\rho$ can be given by [9]

\[
\widetilde{T}[\rho] = \sum_{k=1}^{9} \text{tr} \left[ \frac{1}{3} |v_k^*\rangle \langle v_k| \rho \right] |v_k\rangle \langle v_k|, \tag{3}
\]

with the complete measurement $\{M_k = |v_k^*\rangle \langle v_k| / 3\}_{k=1}^{9}$ where $|v_k^*\rangle$ is the complex conjugation of $|v_k\rangle$ and the vectors $|v_k\rangle$ are normalized and given by

\[
|v_1\rangle = \frac{1}{\sqrt{\omega}} \begin{pmatrix} 1 \\ \omega \\ 0 \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{\omega^2}} \begin{pmatrix} 1 \\ \omega^2 \\ 0 \end{pmatrix}, \quad |v_3\rangle = \frac{1}{\sqrt{\omega^3}} \begin{pmatrix} 1 \\ \omega^3 \\ 0 \end{pmatrix},
\]

\[
|v_4\rangle = \frac{1}{\sqrt{\omega}} \begin{pmatrix} 1 \\ 0 \\ \omega \end{pmatrix}, \quad |v_5\rangle = \frac{1}{\sqrt{\omega^2}} \begin{pmatrix} 1 \\ 0 \\ \omega^2 \end{pmatrix}, \quad |v_6\rangle = \frac{1}{\sqrt{\omega^3}} \begin{pmatrix} 1 \\ 0 \\ \omega^3 \end{pmatrix},
\]

\[
|v_7\rangle = \frac{1}{\sqrt{\omega}} \begin{pmatrix} 0 \\ \omega \\ 1 \end{pmatrix}, \quad |v_8\rangle = \frac{1}{\sqrt{\omega^2}} \begin{pmatrix} 0 \\ \omega^2 \\ 1 \end{pmatrix}, \quad |v_9\rangle = \frac{1}{\sqrt{\omega^3}} \begin{pmatrix} 0 \\ \omega^3 \\ 1 \end{pmatrix},
\]

where $\omega = \exp(i2\pi/3)$ is the relative phase between two occupied bases. Note that these are known as symmetric and informationally complete positive-operator-valued-measures (SIC-POVM) [10]. This clearly shows that one can implement SPA-T for qutrit states by mixing 9 states $\{|v_k\rangle\rangle_{k=1}^{9}$ with probabilities $p_k = \text{tr} [M_k \rho]$ for each state. Hence, to implement SPA-T, one should prepare the state $\{|v_k\rangle\rangle$ according to the measurements outcomes $\text{tr} [M_k \rho]$ for $k = 1, \cdots, 9$. The scheme for implementing SPA-T is graphically shown in Fig. 1 where each operation of the measurement and preparation is denoted as $\widetilde{T}_k$.

Let us now describe the linear optical implementation of the SPA-T shown in Eq. (3). We encode a single-photon qutrit state using the path and polarization of the single photon as shown in Fig. 2 (a): $|0\rangle = |a,H\rangle$, $|1\rangle = |a,V\rangle$, and $|2\rangle = |b,H\rangle$ and the corresponding general qutrit states are described in Fig. 2 (b). A heralded single-photon state is prepared using the spontaneous parametric down-conversion (SPDC) process. A 6-mm-thick type-I $\beta$-BaB$_2$O$_4$ (BBO) crystal was pumped by a 405nm diode laser operating at 100 mW, producing a pair of 810 nm photon pairs (one is called the signal, and the other is called the idler) in the condition of frequency degenerate, non collinear phase matching. Conditioned on the detection of the idler photon, the signal photon is prepared in a single-photon state, i.e. a heralded single-photon state [11]. Then, using a set of wave plates (WP) and a calcite beam displacer (BD), arbitrary single-photon qutrit states $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle$ where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ can be prepared. Before sending the signal photons to the set-up, we observed the initial coincidence count rate between the signal and idler photons of approximately 18,000 Hz using interference filters with full width at half-maximum bandwidth of 10 nm and 5 nm for the signal and idler photons, respectively.

The experimental set-up for realizing SPA-T is shown in Fig. 2 (c) where the whole set-up consists of 4
parts: 1) State Preparation, 2) SPA Measurement, 3) SPA Preparation, and 4) Quantum State Tomography (QST). At the state preparation part, the initial single-photon is transformed into an arbitrary qutrit state, \( \psi = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle \). When the single photon passes through the BD1 which is 4.1 cm thick, the horizontally polarized components are refracted and the vertically polarized components are transmitted without any refraction. The separation between them is 4.0 mm. By making use of the half-wave plate (HWP) in front of the BD1, one can adjust the ratio of the upper path \( a_{1} \) to the lower path \( b_{1} \). The HWP and QWP (quarter-wave plate) behind the BD1 are used for adjusting the ratio of H- and V-polarizations and changing the relative phase differences among three bases.

At the SPA measurement part, the initial state \( \rho \) is projected to one of nine bases \( \{ v_{k}^{\alpha} \} \) with equal probability. Here, the measurement process is the reverse process of the state preparation. Only the photons on the path \( d_{1} \) of the BD2 in the SPA measurement part are post-selected while the photons in the \( c_{1} \) and \( e_{1} \) modes are not detected. Then, the state in the path \( d_{1} \) is projected to the vertically polarized state with probability \( \langle v_{k}^{\alpha} | \rho_{in} | v_{k}^{\alpha} \rangle \) by making use of WPs (HWPs and \( \lambda/3\)-WPs) and a polarizer. \( \lambda/3\)-WP gives the relative phase difference of \( 2\pi/3 \) to the polarization component orthogonal to its fast optic axis with respect to the one parallel to its fast optic axis. By adjusting the angle settings of HWPs and \( \lambda/3\)-WP, one can prepare an arbitrary measurement basis. We used the motorized rotation stages to set the angles of the WP precisely. Notice that since two BDs in the state preparation and SPA measurement parts constitute an interferometer, one should fix the relative phase difference between the upper and lower paths. We can adjust the relative phase difference by horizontally tilting the BD2 and we set the relative phase difference to 0 (modulo \( 2\pi \)). The advantage of using two BDs is that one can make the interferometer very stable. In this experiment, the relative phase difference was maintained over a day.

In detail, for the case of \( k = 1, 2, 3 \), the measurement basis has the equal ratio of \( |0\rangle = |a_{1}, H\rangle \) and \( |1\rangle = |a_{1}, V\rangle \) with different relative phases. At first, for the \( |2\rangle = |b_{1}, H\rangle \) component, we make it pass through the mode \( e_{2} \) since we should not detect it. Meanwhile, the components in the mode \( a_{1}, |0\rangle \) and \( |1\rangle \) components, are projected to the vertically polarized state in the mode \( d_{1} \) with probability \( \langle v_{k}^{\alpha} | \rho_{in} | v_{k}^{\alpha} \rangle \) using WPs in front of BD2. Note that for the case of \( k = 1, 2, 3 \), the projection is already made just after BD2, so we make the state in mode \( d_{1} \) pass through the polarizer without loss. On the other hand, for the case of \( k = 4, 5, 6 \), the projection basis has the equal ratio of \( |1\rangle \) and \( |2\rangle \) with different relative phases. Hence, one makes H-polarized component in path \( a_{1} \) refract at the BD2, so they are recombined at the path \( d_{2} \) and the combined state is projected to the vertically polarized state with probability \( \langle v_{k}^{\alpha} | \rho_{in} | v_{k}^{\alpha} \rangle \).

For the case of \( k = 7, 8, 9 \), the process is
similar to the case of $k = 4, 5, 6$. After the SPA measurement part, once the projection measurement has occurred, the corresponding state $\vert v_k \rangle$ is prepared from the vertically polarized state $\vert V \rangle$ using BD3 and WPs at the SPA preparation part.

Finally, for a given input state $\rho_{in}$, the state applied to experimentally realized SPA-T, $\tilde{T}_{exp} \rho_{in}$ is constructed by the probabilistic sum of nine equally weighted $\tilde{T}_k$ operations. The output state is experimentally identified by quantum state tomography and is then compared to the ideal case, $\tilde{T} \rho_{in}$. By doing QST, we can experimentally reconstruct the density matrices of the quantum states [12, 13]. In general, for $d$-dimensional quantum states, i.e., qudit states, at least $d^2$ different measurements are required to analyze the states. Here, to do QST on a qutrit state, we used $\{|v_k\rangle \langle v_k|\}_{k=1}^9$ as a set of nine QST measurement bases [10, 14]. In addition, in order to confirm that the experimentally implemented SPA-T operation is performed well, we did the quantum process tomography (QPT) [15]. In this experiment, to perform SPA-T, each set of the measurement and preparation is switched every 3 seconds, and the whole experiment is repeated 3 times. QST and QPT results were obtained using the maximum likelihood estimation method [12, 15].

In order to graphically show how SPA-T transforms the quantum state, we show the results for two input states $\rho_{in}^{(1)}$ and $\rho_{in}^{(2)}$ among the variety of input states that we tested. The experimentally prepared input state $\rho_{in}^{(1)}$ is identified by QST and the density matrix of this input state is shown in Fig. 3 (a). The transposed state of the input state can be obtained as $\tilde{T} \rho_{in}^{(1)}$ which is represented in Fig. 3 (b). Notice that the real part of the density matrix $\langle \text{Re} \rho \rangle$ remains without any changes but the off-diagonal terms in the imaginary part of the density matrix $\langle \text{Im} \rho \rangle$ are interchanged. This is because of the Hermitian property of the density matrix. Then, the resultant state $\tilde{T}_{exp} \rho_{in}^{(1)}$ after the experimentally realized SPA-T operation is shown in Fig. 3 (c). Compared to the input state $\rho_{in}^{(1)}$, the amplitudes of the off-diagonal terms in $\tilde{T}_{exp} \rho_{in}^{(1)}$ are decreased and the magnitude differences among diagonal terms are reduced due to the effect of mixing with a white noise. One can also easily notice that the off-diagonal terms in imaginary parts are interchanged like the transposed state. Assuming the ideal SPA-T operation, the output state would be $\tilde{T} \rho_{in}^{(1)}$ and is represented in Fig. 3 (d). To quantify the performance of the realized operation, we evaluate the Uhlmann’s fidelity $F(\rho, \sigma) = \frac{1}{\sqrt{\text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}}}$ between two states, one from the experimental realization $\tilde{T}_{exp} \rho_{in} (\rho)$ and the other from the ideal one $\tilde{T} \rho_{in} (\sigma)$ [16]. Our experimental results show $F(\tilde{T}_{exp} \rho_{in}^{(1)}; \tilde{T} \rho_{in}^{(1)}) \approx 0.999$.

The reconstructed density matrix of the another experimentally prepared input state $\rho_{in}^{(2)}$ is graphically shown in Fig. 4 (a). Like the result for $\rho_{in}^{(1)}$, the density matrices of $\tilde{T} \rho_{in}^{(2)}$, $\tilde{T}_{exp} \rho_{in}^{(2)}$, and $\tilde{T} \rho_{in}^{(2)}$ are represented in Fig. 4 (b), (c), and (d), respectively. Note that although the initial state $\rho_{in}^{(2)}$ has almost no components in $\{|0\rangle\}$ basis, $\tilde{T}_{exp} \rho_{in}^{(2)}$ has $\{|0\rangle\}$ basis component due to the mixing with a white noise. The obtained fidelity value
the ideal and realized operations is \( F(\tilde{T}_{\text{exp}}, \tilde{T}_{\text{exp}}) \approx 0.982 \).

is \( F(\tilde{T}_{\text{exp}} [\rho_{m}^{(2)}], \tilde{T} [\rho_{m}^{(2)}]) \approx 0.999 \). We also repeated SPA-T to a few more states, and the obtained values of fidelities are higher than 0.989.

The realized operation which has been performed in the experiment can be fully identified by QPT. In general, for a quantum operation of \( d \times d \) dimensional quantum systems, \( d^2 \) different input states are required to do QPT and one needs to analyze the state before and after the realized operation by QST for each input state. A quantum process \( \varepsilon \) of a single qudit can generally be expressed as

\[
\varepsilon(\rho) = \sum_{m, n=0}^{d^2-1} \chi_{m,n} \rho_{n,m} \chi_{n,m}^\dagger
\]

where \( \{\chi_m\}_{m=0}^{d^2-1} \) forms the complete set of \( d \times d \) dimensional operator basis and the process matrix \( \chi \) gives the complete characterization of the operation \( \varepsilon \). Note that \( \chi \) is a \( d^2 \times d^2 \) matrix. If one chooses a certain basis set \( \{\chi_m\}_{m=0}^{d^2-1} \), the process matrix \( \chi \) is uniquely determined. By doing QST, \( \chi \) can be experimentally identified. For a single qutrit system, nine operator bases are needed, and we used the following as the complete operator basis

\[
\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

where one can easily notice that \( \lambda_0 \) is an identity operator. The ideal process matrix \( \chi_{\text{ideal}} \) for SPA-T is given as

\[
\chi_{\text{ideal}} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

and the above \( \chi_{\text{ideal}} \) is graphically shown in Fig. 5 (a). The process matrix for the performed operation is experimentally constructed by doing QPT, and the obtained process matrix \( \chi_{\text{exp}} \) is shown in Fig. 5 (b). The average fidelity between the ideal and the realized operation is evaluated as \( F(\tilde{T}, \tilde{T}_{\text{exp}}) = \int d\psi F(\tilde{T} \| \psi \rangle \langle \psi \|, \tilde{T}_{\text{exp}} \| \psi \rangle \langle \psi \|) \approx 0.982 \). Here, the discrepancy between the ideal and realized operation is mainly caused by misalignment of the relative phase difference between two paths of the interferometer consisting of two BDs. For detailed discussions on the gate fidelity related to non-unitary operations, see Reference [17].

We have shown an experimental realization of an approximate transpose for qutrit systems, which is a proof-of-principle for further applications. The scheme is based on the SPA to transpose in the computational basis, and only applies measurement and preparation of quantum states. The realization is shown for qutrit states constructed out of four degrees of freedom which a single photon has in its polarization and path. Our work demonstrates that SPAs for high-dimensional quantum systems can be realized in experiment within the present-day technology.
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