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Scheme for directly observing the noncommutativity of the position and the momentum operators with interference

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Although noncommutativity of a certain set of quantum operators (e.g., creation and annihilation operators and Pauli spin operators) has been shown experimentally in recent years, the commutation relation for the position and the momentum operators has not been directly demonstrated to date. In this paper we propose and analyze an experimental scheme for directly observing the noncommutativity of the position and the momentum operators using single-photon quantum interference. While the scheme is studied for the single-photon state as the input quantum state, the analysis applies equally to matter-wave interference, allowing a direct test of the position-momentum commutation relation with a massive particle.

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I. INTRODUCTION

In quantum physics, a certain set of observables does not commute and this noncommutativity of the conjugate observables leads to the uncertainty relation, which is at the heart of many unique quantum effects [1,2]. It also has been the subject of many illuminating debates on quantum physics [3,4]. For instance, the famous Einstein-Bohr debate was on the uncertainty principle regarding the position and momentum measurement as, in quantum physics, two noncommuting observables cannot be measured accurately simultaneously [4]. Moreover, Einstein-Podolsky-Rosen argued against such an apparent lack of *simultaneous physical reality* in their famous 1935 paper [5].

Although the commutation relation has been well established theoretically since Heisenberg introduced the canonical commutation relation of the position and the momentum operators, experimental tests on the noncommutativity of conjugate operators have been rather scarce. The noncommutativity of Pauli spin operators σ_x, σ_y, and σ_z has been demonstrated with fermions (neutrons) [6,7] and recently with bosons (photons) [8,9]. Also, the noncommutativity of bosonic creation $\hat{a}^\dagger$ and annihilation operators $\hat{a}$ has recently been demonstrated with photons [10,11]. However, the noncommutativity between the position $\hat{x}$ and the momentum $\hat{p}$ operators has always been associated with the uncertainty principle regarding the position and the momentum measurements of a particle, as pictured in the Heisenberg microscope [12,13]. Note though that the single-particle uncertainty relation breaks down for the position-momentum entangled two-particle system as discussed in Ref. [5] and demonstrated in Refs. [14,15]. The position-momentum uncertainty relation can be investigated with the single-slit diffraction experiment involving a quantum object [12,13] and it has been demonstrated experimentally for neutrons [16] and for large fullerene molecules [17]. It is nevertheless interesting to point out that the noncommutativity relation for the position and the momentum operators itself has not been directly observed to date as was demonstrated for the Pauli spin operators and the bosonic creation and annihilation operators.

In the present work we propose and analyze an experimental scheme to directly observe the noncommutativity of the position and the momentum operators using the transverse spatial degree of freedom x of a single-photon wave function $\psi(x)$ [in the sense that $|\psi(x)|^2$ gives the probability distribution] [18]. The position and the momentum operators $\hat{x}$ and $\hat{p}$ are implemented using position-dependent attenuators, phase plates, and lenses. The commutator and the anticommutator of the position and the momentum operators are constructed with single-photon quantum interference. For an initial Gaussian wave function $\psi(x)$, we find that applying the commutator leaves the state unchanged, whereas applying the anticommutator results in a Wigner function with negativity, starkly different from the Wigner function of the initial wave function [19]. Finally, we discuss how the proposed scheme can be applied to matter-wave interferometry to directly observe the noncommutativity of the position and the momentum operators for a particle with mass or a macroscopic quantum state of matter.

II. IMPLEMENTING $\hat{x}$-$\hat{p}$ COMMUTATION OPERATIONS

Consider a quasimonochromatic single photon traveling in the z direction. Since the Hilbert space that represents the transverse spatial degrees of freedom of the photon is isomorphic to the Hilbert space that describes the quantum state of a point particle in two dimensions [20–22], we may use the quantum-wave-function formalism for a point particle to describe the transverse spatial wave function of a single photon. We choose to describe only one transverse spatial degree of freedom, namely, the transverse position of the photon x, without loss of generality due to the orthogonality.

In the position basis, the relation $\hat{x}|x\rangle = x|x\rangle$ holds, so an arbitrary pure state $|\alpha\rangle$ can be written as $|\alpha\rangle = \int dx\, \psi_\alpha(x)|x\rangle$, where $\psi_\alpha(x)$ is the transverse spatial wave function for the state $|\alpha\rangle$. In the conjugate momentum basis, the relation $\hat{p}|p\rangle = p|p\rangle$ holds, so $|\alpha\rangle = \int dp\, \phi_\alpha(p)|p\rangle$, where $\phi_\alpha(p)$ is the corresponding wave function for $|\alpha\rangle$ in this basis.

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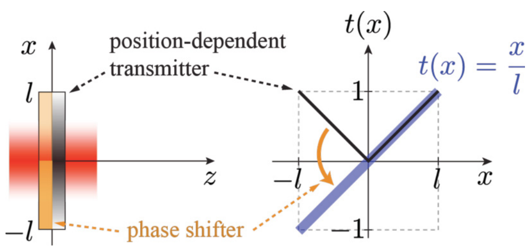

FIG. 1. (Color online) Scheme for implementing the $\hat{x}$ operation. Applying $\hat{x}$ to the wave function introduces the amplitude transmission coefficient $t(x) = x/l$. The phase shifter introduces the relative phase shift of π for the region $x \in [-l,0]$. The position-dependent transmitter introduces the linear amplitude transmission coefficient $t(x) = |x|/l$.

To implement a quantum operator is to find a quantum operation that results in the desired output quantum state. For the position operator $\hat{x}$,

$$\hat{x}|\alpha\rangle = \hat{x}\left(\int dx\, \psi_\alpha(x)|x\rangle\right) = \int dx\, x\psi_\alpha(x)|x\rangle, \quad (1)$$

which means that the action of $\hat{x}$ to the wave function $\psi_\alpha(x)$ is multiplication of x to the wave function, i.e., $x\psi_\alpha(x)$. Then the corresponding operation is to introduce the amplitude transmission coefficient $t = x/l$, which can be implemented by using a set that consists of a π phase shifter followed by an attenuator with the linear amplitude transmission coefficient $t(x) = |x|/l$ (see Fig. 1). The phase shifter introduces the relative phase shift of π for the region $x \in [-l,0]$ with respect to the region $x \in [0,l]$. The phase shifter can be implemented, for instance, with a piece of glass by polishing away thickness corresponding to $\lambda/2$ for the region $x \in [-l,0]$, where λ is the central wavelength of the photon. The phase shifter and the attenuator together achieve the overall amplitude transmission coefficient of $t(x) = x/l$, implementing the operation corresponding to a dimensionless position operator $\tilde{x} = \hat{x}/l$.

For the momentum operator $\hat{p}$ we have a similar result

$$\hat{p}|\alpha\rangle = \hat{p}\left(\int dp\, \phi_\alpha(p)|p\rangle\right) = \int dp\, p\phi_\alpha(p)|p\rangle. \quad (2)$$

To implement the quantum operation that results in $p\phi_\alpha(p)$, we simply need to place the set consisting of the phase shifter and the attenuator in Fig. 1 at the Fourier plane of the incoming photon. This is easy to see by simply reexpressing Eq. (2) in the position basis $\hat{p}|\alpha\rangle = \int dp\, p\phi_\alpha(p)|p\rangle = \int dx \langle x|\hat{p}|\alpha\rangle|x\rangle$. The scheme for implementing the $\hat{p}$ operation is shown in

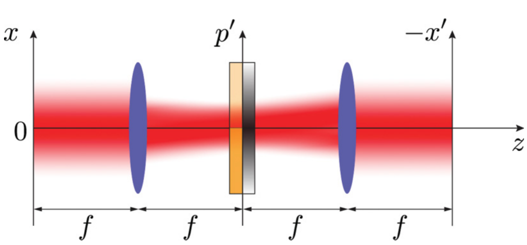

FIG. 2. (Color online) Scheme for implementing the $\hat{p}$ operation. Two lenses of the same focus f make a $4f$ imaging system with the $\hat{x}$ operation implemented at the Fourier plane.

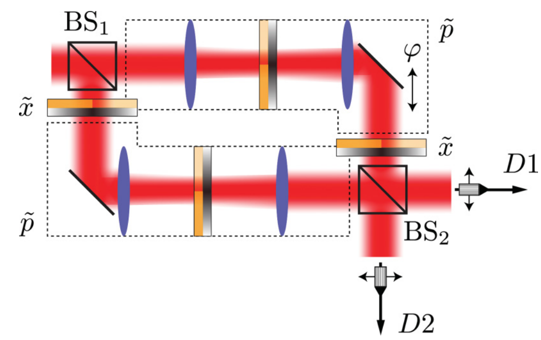

FIG. 3. (Color online) Coherent superposition of two quantum operations $\tilde{x}\tilde{p}$ and $\tilde{p}\tilde{x}$ is accomplished with a Mach-Zehnder interferometer. Single-mode fiber tips (connected to single-photon detectors) are scanned in transverse directions to measured the output states.

Fig. 2. Two lenses of the same focus f form a $4f$ imaging system and the $\hat{x}$ operation is implemented at the Fourier plane of the $4f$ imaging system.

To be more specific, consider the scheme in Fig. 2 and assume that the wave function in the input plane ($z = 0$) is given as $\psi_\alpha(x)$. The first lens maps the position-space wave function $\psi_\alpha(x)$ onto the momentum-space wave function $\phi'_\alpha(p')$ [23],

$$\phi'_\alpha(p') = \frac{1}{\sqrt{i\lambda f}} \int \psi_\alpha(x) \exp\left(-i\frac{2\pi}{\lambda f}xp'\right) dx, \quad (3)$$

where $p' = \frac{\lambda f}{2\pi\hbar}p$. At the Fourier plane, the set of the phase shifter and the attenuator performs the transformation $\phi'_\alpha(p') \to p'\phi'_\alpha(p')/l$. Finally, at the output of the optical system, i.e., $z = 4f$, the wave function is now given as

$$\frac{1}{\sqrt{i\lambda f}} \int \frac{p'}{l}\phi'_\alpha(p') \exp\left(i\frac{2\pi}{\lambda f}x'p'\right) dp' = -i\langle x'|\tilde{p}|\alpha\rangle, \quad (4)$$

where $\tilde{p} = \frac{\lambda f}{2\pi\hbar l}\hat{p}$ is the dimensionless momentum operator. Note that we chose $x' = -x$ to account for the inverting nature of a $4f$ imaging system.

We have so far discussed how to implement the dimensionless position $\tilde{x}$ and momentum $\tilde{p}$ operators. Let us now discuss how to probe the commutation relations for the operators by using single-photon interference. Consider the experimental scheme shown in Fig. 3. A single-photon state enters the Mach-Zehnder interferometer via the beam splitter at the left. At each arm of the interferometer, optical systems that implement the dimensionless position and momentum operators are placed, but in different order. In the top path $\tilde{p}\tilde{x}$ is implemented and in the bottom path $\tilde{x}\tilde{p}$ is implemented. The two quantum operations are then coherently superposed at the second beam splitter with a relative phase φ [10,11]. The quantum superposition of operators is implemented at the output of the Mach-Zehnder interferometer, namely, $\tilde{x}\tilde{p} + e^{i\varphi}\tilde{p}\tilde{x}$ at D_1 and $\tilde{x}\tilde{p} - e^{i\varphi}\tilde{p}\tilde{x}$ at D_2. The resulting wave functions can be analyzed by measuring the single-photon detection probabilities with scanning fiber tips in the transverse direction.

III. RESULTS AND DISCUSSION

Suppose now that a single-photon with the wave function $\psi(x)$ enters the interferometer and the relative phase is set at $\varphi = \pi$. At D_1 we have $[\tilde{x},\tilde{p}]\psi(x) = \frac{C}{\hbar}[\hat{x},\hat{p}]\psi(x)$, where $C \equiv \frac{\lambda f}{2\pi l^2}$. Since $[\hat{x},\hat{p}] = i\hbar$, the commutator for the dimensionless operators becomes $[\tilde{x},\tilde{p}] = iC$, so the commutator operated on the wave function leaves the state unchanged (after normalization). Similarly, at D_2 the anticommutator for the dimensionless operators acting on the wave function results in $\{\tilde{x},\tilde{p}\}\psi(x) = \frac{C}{\hbar}\{\hat{x},\hat{p}\}\psi(x)$, which means that the resulting wave function appearing at D_2 is different from the input wave function. Note that the commutator and the anticommutator output ports, set at D_1 and D_2, respectively, can be switched by choosing the relative phase value $\varphi = 0$.

As an example let us consider a quasimonochromatic single-photon state with a Gaussian wave function

$$\psi(x) = \sqrt[4]{\frac{2}{\pi w^2}}\exp(-x^2/w^2) \tag{5}$$

at the input port of the Mach-Zehnder interferometer. The detection probability $I^1_\varphi(x)$ at D_1 as a function of x and φ is shown in Fig. 4. Here we assume $\lambda = 800$ nm, $w = 0.5$ mm, $l = 1.5$ mm, and $f = 50$ cm. The figure shows that at $\varphi = \pi$, which corresponds to the commutator case, the Gaussian input wave function is reproduced at the output as expected. In contrast, at $\varphi = 0$, which corresponds to the anticommutator case, the output shows interference fringes along the x direction.

A complete characterization of the spatial coherence of the wave function $\psi(x)$, however, requires tomographic reconstruction of the spatial Wigner function $W(x,p)$, where x and p refer to the actual position and momentum of the single photon. The spatial Wigner function $W(x,p)$ can be reconstructed by employing, for example, an area-integrated detection scheme [19]. In Fig. 5 the spatial Wigner functions for the input wave function and the two output wave functions (i.e., the commutator-operated wave function and the anti-commutator-operated wave functions) are shown. The figure clearly shows that the input and the commutator-operated wave functions have the same Wigner functions. In other words, it shows that the quantum operation corresponding to the $\hat{x}$ and $\hat{p}$ commutator is indeed equivalent to (a constant multiple of) the identity operation. In contrast, the anti-commutator-operated wave function exhibits a completely different Wigner function, interestingly with a clear signature of negativity.

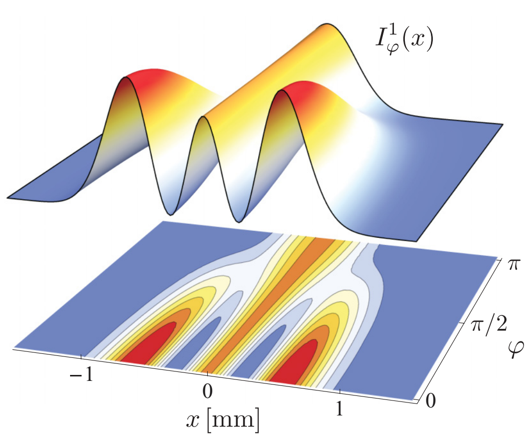

FIG. 4. (Color online) Probability distribution $I^1_\varphi(x)$ at D_1 calculated as a function of φ and x assuming a Gaussian input wave function. The commutator $\varphi = \pi$ acting on the input Gaussian wave function leaves the wave function unchanged. The anticommutator $\varphi = 0$, however, causes the input wave function to change, resulting in interference.

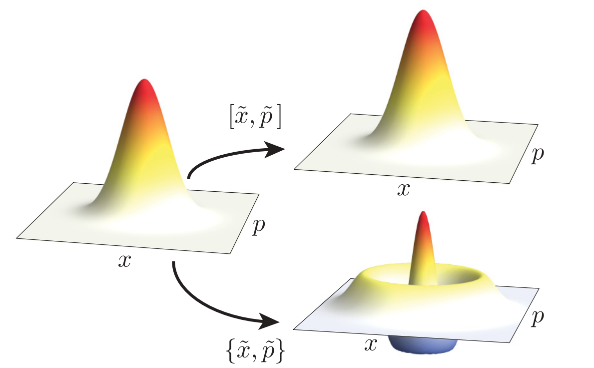

FIG. 5. (Color online) Spatial Wigner functions $W(x,p)$ of the input wave function $\psi(x)$ (Gaussian, everywhere positive) and the states after the commutator $[\tilde{x},\tilde{p}]$ and anticommutator $\{\tilde{x},\tilde{p}\}$. Applying the commutator to the wave function results in the identical spatial Wigner function as the input, whereas applying the anticommutator lead to a starkly different Wigner function with negativity.

IV. CONCLUSION AND OUTLOOK

The commutation relation for the position and the momentum operators leads to the position-momentum uncertainty relation and, in experiment, the position-momentum uncertainty relation has been demonstrated in single-slit diffraction experiments involving a massive (neutrons, fullerene, etc.) or a massless particle (photons) [13,16,17]. In this work we have proposed and analyzed an interferometric scheme for directly (i.e., without involving the uncertainty relation) observing the commutation relation for the position and the momentum operators using single-photon quantum interference. The proposed scheme requires only linear optical elements and single-photon detectors so it should be possible to realize such an experiment if the position-dependent attenuator and the π phase shifter can be precisely engineered. In practice, a soft-edge graduated neutral density filter (whose amplitude transmission coefficient is linear) can function as a position-dependent attenuator and a molding technique can be used for producing a precise phase shifter [24].

Although the interferometric scheme proposed in this paper is focused on single-photon interferometry, the proposed concept can readily be expanded to matter-wave interferometry involving a massive particle or even the macroscopic quantum state of matter. For instance, essential elements in the proposed scheme can be built for a Bose-Einstein condensate, a macroscopic quantum object. Position-dependent attenuators can be built using the quantum tunneling effect through a laser-induced potential barrier [25], focusing of a matter wave can be accomplished by using light as an atomic lens [26,27], and a matter-wave interferometer can be built by using the sequential Bragg momentum transfer effect [28]. Moreover, it is possible to reconstruct the spatial Wigner function for a massive particle [29], thus making a direct experimental test of the position-momentum commutation relation with a macroscopic quantum object within the reach of present-day technology.

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